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OSCILLATORY GRAVITY WAVES IN FLOWING WATER

Turgut Sarpkaya,[1] J. M. ASCE

ABSTRACT

The main objectives of this study are to determine the conditions of stability of progressive oscillatory gravity waves in water flowing over smooth and artifically roughened surfaces and to evaluate the combined effect of various interrelated characteristics of flowing water, specifically those of viscous shear, turbulent mixing shear, and nonuniform velocity distribution, on their propagation.

At present, there are neither experimental nor conclusive theoretical results which constitute a solution to the problem under consideration. It thus becomes highly desirable to investigate the conditions of neutral stability of progressive oscillatory gravity waves in flowing water and to evaluate the effect of various characteristics of flowing water on the velocity of propagation of stable waves by a comparison of the experimental results with those obtained from the available theories, particularly those of Stokes and Struik.

The period and various geometrical characteristics of stable ascendant oscillatory gravity waves in water flowing in a smooth and an artificially roughened rectangular channel were measured for a wide range of depth and velocity of flow and suitable settings of wave generator. The following results were obtained:

(1) For a given flow, there is only one neutrally stable wave. All the other waves for this flow show either an exponential damping or an amplification during their propagation along the channel. It is readily possible to separate the two regions by the surface of neutral stability which was determined experimentally and presented graphically.

(2) The celerity of a stable ascendant wave in flowing water is smaller than that obtained from the theories of Stokes and Struik. The deviation of the measured wave velocity from that obtained from potential theory increases with increasing Froude numbers and decreases with increasing Reynolds numbers.

(3) For a channel with an artificially roughened bottom, the deviation is greater than the corresponding one in a channel with a smooth bottom.

(4) The experimentally determined profiles of stable oscillatory gravity waves vary about the theoretical profiles as described by Struik.

INTRODUCTION

The exact knowledge of the characteristics of waves in flowing water is extremely important not only from the scientific point of view but also from the practical one. It is particularly important in connection with the

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penetration of waves into an estuary and probably with the determination of scour in the river bed by either ascendant or descendant waves generated by wind. The analysis of all these problems requires, first of all, a knowledge of the laws governing wave movement in a nonerodible channel.

Until the last two decades, knowledge of the motion of oscillatory gravity waves in water was based largely on theoretical studies. Several rigorous solutions, differing chiefly in their results for the motion of the water particles involved, were obtained[1, 2, 3, 4].

Experimental confirmation of these theoretical solutions has been the main purpose of recent studies. For the waves in still water, these investigations[5, 6, 7] have shown that aside from mass transport the magnitude of the difference in value of the wave characteristics defined by the theories of rotational and irrotational wave motion is smaller than the experimental error. If the waves are generated in a liquid which is already in motion some difference would be expected between observations and classical theories resulting from the effect of various characteristics of flowing water, specifically those of viscous shear, turbulent mixing shear, and nonuniform velocity distribution which are definitely interrelated. In the one hand, viscous action in the boundary layer produces vorticity which diffuses through the fluid and modifies the whole velocity distribution. The subsequent wave motion thereby inherits a rotational motion, and the usual assumption of a velocity potential becomes inadequate. On the other hand, turbulence becomes an important factor and it plays a role that may even overshadow that of gravity. Furthermore, both the velocity distribution and the turbulence structure of the primary flow are completely altered by the wave motion. In addition to these, perhaps one of the most complicating factors is that the waves generated in flowing water are not always neutrally stable. Some of them are damped and some of them are amplified according to the existing flow conditions. Therefore, the effect of the various factors mentioned previously on the propagation of waves in flowing water can be evaluated, but only after the determination of the conditions of neutral stability.

In this study, an attempt has been made to establish the conditions of stability of ascendant waves in flowing water and to evaluate the effect of the various characteristics of flow on their propagation.

Previous Investigations

Certain phases of the general problem of the effect of various characteristics of the primary flow on the velocity of propagation of disturbances have been studied previously. The problem is, however, so complex that none of these phases has been completely solved.

Bazin's experiments[8] on the propagation of a solitary wave were the first to demonstrate the incongruity between observations and the theory despite the good agreement in still water. The effect of the nonuniform velocity distribution on translation waves has been studied in great detail by Boussinesq.[9] Favre[10] has employed the Boussinesq integrals to describe the motion of waves occuring during the operation of navigation locks. More than half a century after Boussinesq's analysis was completed, Keulegan and Patterson presented an analysis of the combined effects of turbulent resistance and inertia on the motion of flood waves and the other eaves of translation in open channels[11]. All of these investigations were, it should be remembered, carried out for translatory gravity waves in flowing water.

Hough[12] presented a solution of certain problems illustrative of the effects of viscosity on the motion of the sea, and concluded that the velocity of wave propagation is slightly decreased by resistance of the ocean bed, by the dissipation of energy through internal viscosity, and by turbulence. Abdulla[13] made an attempt towards a solution of wave motion superposed on a current which is caused by winds. The current is assumed to follow Ekman's spiral law in speed but to remain constant in direction. From the comparison of his results with gravitational waves, Abdullah found that the total effect of turbulence and gravity is to give the waves a velocity which is smaller than that provided by gravity alone. Biésel[14], has presented a theoretical analysis for the effect of nonuniform velocity distribution on the velocity of propagation of small amplitude oscillatory gravity waves in open channels. This analysis will be discussed later in detail as it partially confirms the experimental results.

For very long waves, that is for very large values of $\alpha L^2/d^3$, in the linearized theory Burns[15] obtained a general equation connecting the wave velocity with the velocity in the undisturbed stream and solved the equation in several special cases. He has shown generally that the velocity of a long wave relative to the mean flow is always greater than the value given by the classical theory. In the appendix of Burns' paper, Lighthill examined the "critical Froude number" above which no ascendant wave propagation is possible. On the basis of some assumptions which will be discussed later, Lighthill has found that the nonuniformity in the velocity distribution has opposite effect for large and small wave lengths. For Stokian waves where the wave motion is predominantly at the surface, the relatively high fluid velocity in this region causes the critical Froude number based on the mean velocity to be lower. But for really long waves, in which the motion is almost purely longitudinal, the decreased velocities near the bottom make a bigger difference than do the increased velocities near the surface.

Considering the second part of the present investigation, one finds that no previous theoretical and experimental study has been devoted to the determination of the conditions of stability of an oscillatory gravity wave. Because of the challenging nature of the stability problem, however, many investigations since the time of Boussinesq have been made for the stability of interfacial gravity waves, for the stability of parallel flows, and for the stability of dunes in the stream beds. Korteweg and de Vries[16] have shown that relatively long waves have a tendency to steepen ahead of their peaks, at a rate proportional to α/d which is resisted by an opposing tendency, independent of amplitude, and due to the departures from uniformity in the distribution of velocity with depth. Korteweg and de Vries have also shown that there is an infinite sequence of wave forms, each such that for one particular ratio of α/d to d^2/L^2 the two tendencies to change of form cancel out, leading to a wave of permanent type. These waves are cnoidal; more precisely, they are graphs of the square of the Jacobian elliptic function cnx for different moduli k_1; one extreme ($k_1 = 0$) is the sinusoidal wave, which occurs when the ratio $\alpha L^2/d^3$ is very small, and at the other extreme ($k_1 = 1$), the wave length becomes large and the waves take on more and more the character of a sequence of solitary waves.

For the stability of an elementary wave, starting from a uniform regime, the evolution with time of a sinusoidal undulation of very small amplitude was studied by H. Jeffreys, and depending upon the case, an exponential damping or amplification was found. Recently the stability of two dimensional small disturbances in laminar flow on a plane boundary was examined by Yih[17].

It was found that for each slope of the channel bottom, a hyperbolic type of curve with the arms extending in the directions of wave number $2\pi/L$ and the Reynolds number separates the stable region from the unstable one.

As has already been mentioned, however, no previous theoretical and experimental investigation exists concerning the conditions of stability of an oscillatory gravity wave of finite amplitude in flowing water. It thus becomes extremely desirable to establish the conditions of stability of waves under consideration.

Theoretical Considerations

The present status of turbulence theory and the mathematical techniques seem to offer little hope that either the effect of turbulence and viscosity or the conditions of stability can be directly determined. In general, an oscillatory gravity wave is unsteady and involves both molar and molecular resistance. Besides the complexity of wave motion itself, the characteristics of turbulence, such as velocity and pressure fluctuations, cannot be asserted a priori. However, neglecting the effect of turbulence and considering only the effect of viscosity with a nonuniform velocity distribution over the cross section and using the Sommerfeld-Orr equation[18], the problem of stability can be formulated conveniently. Despite its simple formulation, the task of evaluating a rigorous solution of the equations obtained for even a simple velocity distribution would be extremely difficult and no attempt has been made at such an evaluation.

For very large Reynolds numbers, the theoretical as well as the practical value of this solution seems questionable. As the Reynolds number increases beyond the critical range, the eddy viscosity becomes increasingly important. In other words, if the Reynolds number is high, the viscous stresses in terms of the mean velocity gradients are negligible in comparison with the apparent stresses due to momentum transport.

Despite the difficulties cited above, the effect of nonuniform velocity distribution alone on the propagation of stable gravity waves of small amplitude can be obtained analytically. In accordance with the above considerations, Biesel[14] neglected the viscous terms in Sommerfeld-Orr equation and made the wave motion steady by fixing the axes on the wave profile. He further assumed a linear velocity distribution given by the expression,

$$C = C_0(1+ky) \qquad 1$$

and obtained the velocity of an ascendant wave to be

$$V_{as} = \sqrt{\left[\frac{U_0-U_1}{2}\frac{\tanh md}{md}\right]^2 + \left[g\frac{\tanh md}{m}\right]} - \left[U_0 - \frac{U_0-U_1}{2}\frac{\tanh md}{md}\right] \qquad 2$$

in which U_o and U_1 represent the absolute surface and bottom velocities of flow, respectively.

Were this velocity obtained from the superposition of a small-amplitude wave velocity on the mean velocity of flow, one would obtain

$$V'_{as} = \sqrt{g\frac{\tanh md}{m}} - \frac{U_0+U_1}{2} \qquad 3$$

If V_t and V_m represent respectively the theoretical and measured wave velocities, the deviation ζ can be defined by the relationship

$$\zeta = \frac{V_t - V_m}{V_m} \qquad 4$$

If Eq. (2) is used for measured velocities and Eq. (3) for theoretical velocities, the deviation ζ can be obtained as

$$\zeta = \frac{\sqrt{g\frac{\tanh md}{m}} - \frac{U_0+U_1}{2}}{\sqrt{\left\{\frac{U_0-U_1}{2}\frac{\tanh md}{md}\right\}^2 + g\frac{\tanh md}{m}} - \left\{U_0 - \frac{U_0-U_1}{2}\frac{\tanh md}{md}\right\}} - 1 \qquad 5$$

The results obtained from this equation and their comparison with the experimental results will be discussed later in detail as it partially confirms the experimental results.

Next we turn our attention to the breaking of waves in flowing water. One of the most important characteristic features of oscillatory gravity waves in flowing water is the continuous deformation of wave shape and the subsequent breaking of waves under the influence of somewhat different factors than those encountered in still water. The limiting condition beyond which no wave propagation exists is determined primarily from the consideration that the wave becomes unstable and breaks when the forward velocity of the water particles at the crest exceeds the velocity of the wave. In still water this in general may be caused by the shoaling of the depth accompanied by the complicated effects of backwash from previously broken waves or by yet little understood action of an opposing wind. It has been known theoretically for quite some time that the asymptotic limit for waves in deep water is $\alpha/L = 0.14$. In fairly shallow waters, it has been pointed out already by various authors that a good approximation is given by the solitary wave theory despite the fact that it is not confirmed by the experiments conducted at M. I. T., in sloping channels. For intermediate conditions in medium depth, the limiting wave steepness is given by Miche[4] as

$$\frac{a}{L} = 0.14 \tanh \frac{2\pi d}{L} \qquad 6$$

Experimental results reported by Danel[19] come very close to this theoretical limiting curve.

In flowing water, however, in addition to the effect of the slope of the channel bottom which does not cause any shoaling, the retardation of the advance of the forward face of the wave by the opposing current and consequent overtaking of the forward face by the remainder of the wave becomes a predominant factor. Evidently, the mean velocity of flow as well as the particular velocity distribution plays an important role in the determination of limiting condition. Therefore, the limiting wave steepness is further related to the Froude and Reynolds numbers of the primary flow in addition to the ratio d/L.

The consideration of all these parameters, however, in the theoretical determination of the limiting condition is as difficult as the evaluation of the stability conditions. It is essential to make simplifying assumptions in order to get some insight into the phenomenon. Lighthill[15] made the assumption that a solution exists with V = 0, corresponding to a stationary wave, above which no upstream propagation is possible. The consideration of the equation

of motion for a stationary wave of sufficiently small amplitude with the particular velocity distribution

$$u = u_1 (y/d)^{\varepsilon} \qquad 7$$

and with the necessary boundary conditions leads to the solution for the critical Froude number as follows:

$$F_c = \frac{\bar{U}}{\sqrt{gd}} = \sqrt{\frac{L}{2\pi d} \frac{1}{(1+\varepsilon)^2} \frac{I_{\frac{1}{2}-\varepsilon}\left(\frac{2\pi d}{L}\right)}{I_{\frac{1}{2}-\varepsilon}\left(\frac{2\pi d}{L}\right)}} \qquad 8$$

To show the effect of nonuniformity in velocity distribution, if the ratio of this expression to the classical expression (its limit as $\epsilon \to 0$) for uniform flow is taken, one finds

$$\frac{F_c}{F} = \sqrt{\frac{1}{\operatorname{Tanh}\frac{2\pi d}{L}} \frac{1}{(1+\varepsilon)^2} \frac{I_{\frac{1}{2}-\varepsilon}(2\pi d/L)}{I_{-\frac{1}{2}-\varepsilon}(2\pi d/L)}} \qquad 9$$

The fundamental assumption made by Lighthill in the derivation of the foregoing equations is not, however, fully acceptable. For the wave propagation becomes impossible with the complete breaking of the wave when the velocity of flow becomes equal to the celerity of energy transmission, which is in general smaller than the wave velocity except for the case of anomalous dispersion. For the waves of small amplitude, the wave energy is transmitted by the group velocity C_g

$$\frac{C_g}{\sqrt{gd}} = \frac{1}{2}\left\{1 + \frac{4\pi d/L}{\operatorname{Sinh} 4\pi d/L}\right\}\sqrt{\frac{L}{2\pi d} \operatorname{Tanh}\frac{2\pi d}{L}} \qquad 10$$

The validity of the foregoing is limited, strictly speaking, to the waves of sufficiently small amplitude. The assumption is made that the energy transmission for the waves under consideration is of the same nature as that of the small amplitude theory. Although, the effect of Froude and Reynolds numbers can be included with the help of the deviation ζ, for the sake of simplicity, this will be omitted.

If the assumption is made that no wave propagation is possible when the celerity of energy transmission is equal to the mean velocity of flow, the critical Froude number becomes

$$F = \frac{1}{2}\left\{1 + \frac{4\pi d/L}{\operatorname{Sinh} 4\pi d/L}\right\}\sqrt{\frac{L}{2\pi d} \operatorname{Tanh}\frac{2\pi d}{L}} \qquad 11$$

It can easily be shown that the condition of the equality of the mean velocity of flow to the celerity of energy transmission corresponds precisely to the equality of the wave lengths of the two ascendant waves of the same period propagating in the same flow. Here it should be noticed that, in general there are two different waves of the same period propagating in the same flow.

Dimensional Analysis

The basic variables pertinent to this phenomenon are: Amplitude, length and velocity of a stable wave; dynamic viscosity and density of the fluid; depth

and mean velocity of flow; shape factor of the flume; spacing and the dimensions of roughness elements; and the acceleration of gravity. These variables yield the following functional relationships:

$$\begin{bmatrix} \frac{a}{d} \\ \frac{L}{d} \\ \frac{V_m}{\bar{U}} \end{bmatrix} = \underset{i-3}{f}\left\{\frac{\bar{U}}{\sqrt{gd}}, \frac{\bar{U}d}{\nu}, \frac{k}{d}, \frac{\lambda}{d}, \beta\right\} \qquad 12$$

To make the comparison of the wave velocities obtained experimentally with those obtained from the theory of irrotational motion, it becomes necessary to introduce the theoretical wave velocity V_t into the analysis. For an ascendant wave, V_t is given by the relation

$$V_t = C - \bar{U} \qquad 13$$

If both sides of this equation are divided by V_m and the numerator and the denominator of the right side by $\sqrt{gd}$ also, one obtains

$$\frac{V_t}{V_m} = \frac{\frac{C}{\sqrt{gd}} - \frac{\bar{U}}{\sqrt{gd}}}{\frac{V_m}{\bar{U}} \frac{\bar{U}}{\sqrt{gd}}} \qquad 14$$

Since $C/\sqrt{gd}$ is a function of only α/d and L/d and in turn a function of only the independent dimensionless parameters, $V_m/\bar{U}$ in Eq. (12) can be replaced by V_t/V_m or by the deviation ζ. If one replaces α/d by α/L and L/d by d/L and also d/k by k/d for convenience, the relationships take the form

$$\begin{bmatrix} \frac{a}{L} \\ \frac{d}{L} \\ \zeta \end{bmatrix} = \underset{i-3}{g}\left\{\frac{\bar{U}}{\sqrt{gd}}, \frac{\bar{U}d}{\nu}, \frac{k}{d}, \frac{\lambda}{d}, \beta\right\} \qquad 15$$

The dependent dimensionless parameters on the left side of the equations represent respectively the wave steepness, the ratio d/L, and the deviation. The parameters on the right side of the equations represent, in the order of their appearance, the Froude number, the Reynolds number, the relative roughness, the relative spacing of the roughness elements, and a channel shape factor.

The consideration of all these variables in the determination of the stability conditions and in the evaluation of the deviation ζ is almost impossible. A useful result can be attained only in isolated instances and then only if all but the most essential parameters are ignored. The existing experimental data and the theoretical knowledge are not sufficient to determine the characteristics of the shape factor β. Furthermore, if the sides and bottom of the channel are smooth, d/k and λ/k need not be considered. Thus, for smooth channels of rectangular cross section, the relationships take the forms,

$$\begin{bmatrix} \frac{a}{L} \\ \frac{d}{L} \\ \zeta \end{bmatrix} = \mathcal{H}_{1\text{-}3}\{F, R\} \qquad 16$$

For the case in which the channel bottom is artificially roughened,

$$\begin{bmatrix} \frac{a}{L} \\ \frac{d}{L} \\ \zeta \end{bmatrix} = \mathcal{H}_{1\text{-}3}\left\{F, R, \frac{d}{k}, \frac{\lambda}{k}\right\} \qquad 17$$

In the present investigation, only one type of strip roughness is used; i. e., λ/k is constant. Therefore:

$$\begin{bmatrix} \frac{a}{L} \\ \frac{d}{L} \\ \zeta \end{bmatrix} = \mathcal{H}_{1\text{-}3}\left\{F, R, \frac{d}{k}\right\} \qquad 18$$

The experimental results will be evaluated according to Eqs. (16) and (18).

Apparatus and Procedure

Wave Flume

The glass-walled flume used in these experiments has a rectangular working section 85 feet long, 2 feet wide, and 1.5 feet deep. Depth and velocity measuring devices and probes for the wave recorder with necessary electrical circuits were mounted on two aluminum carriages which run on two parallel steel rails attached to the top of the walls of the flume.

Artificial roughness consisted of 1/4" x 1/4" continuous brass strips, placed at 1" intervals on the steel bottom of the flume perpendicular to the longitudinal axis of the flume. The relative spacing λ/k of the strips was 4. The depth of the flow was measured to the middle of the height of the strips, i. e., 0.011' was subtracted from the depth measured to the smooth bottom. Actually, the true depth would be somewhere between this depth and the depth to the top of the brass strips.

Water was supplied from the circulation system of the laboratory, and the discharge was measured by means of a calibrated venturi meter. In addition, a triangular weir, which can be conveniently placed at the end of the flume, was used to measure small discharges.

Wave Generator

It was necessary, to design a new wave generator which must at once allow the passage of flow, produce as nearly as possible the water particle motion

in accordance with the theory under consideration, cause a minimum amount of additional turbulence, and be simply constructed and easily operated.

The wave generator developed in the Iowa Institute of Hydraulic Research and used throughout the experiments is principally a combination of Ransford's classical three-bar linkage or ballistic pendulum with a system similar to a venetian blind which is made of metal slats that can be set together at any angle to regulate the passage of water only in one direction (Fig. 1). The use of metal slats to allow the passage of flow was an idea originated at the St. Anthony Falls Hydraulic Laboratory. The venetian-blind type of wave generator adequately produces in varying depths of flow the waves under investigation throughout the complete range of study and is, moreover, very simple to operate.

In order to eliminate wave reflections from the end of the flume opposite the generator, it was necessary to install wave dampers across the section. A basket made of No. 32 wire screen was filled with 1/4 and 1/2 inch gravel. The basket was hung on two parallel steel rails at a slope of approximately 30 in the direction of flow. The elevation was so adjusted that the bottom of the screen basket was a half wave height below the mean elevation of the water surface. This, with one or two wire screens stretched below the section, provided excellent wave absorption.

Experimental Procedure

The amplitude, length, and shape of waves were determined from photographs. Readings were made to thousandths of a foot under the microscope with no difficulty.

Wave periods were obtained from recordings made with a Sanborn Twin-Viso recorder. Wave velocities were also obtained by placing two parallel wire gages 10 feet apart in the wave tank and recording the wave profile from these gages on the Sanborn Twin-Viso recorder. A timing device made four marks per second on the recorded permapaper. Thus, knowing the distance between the gages and the time required for the wave to travel from one gage to the next, the wave velocity between the gages was determined.

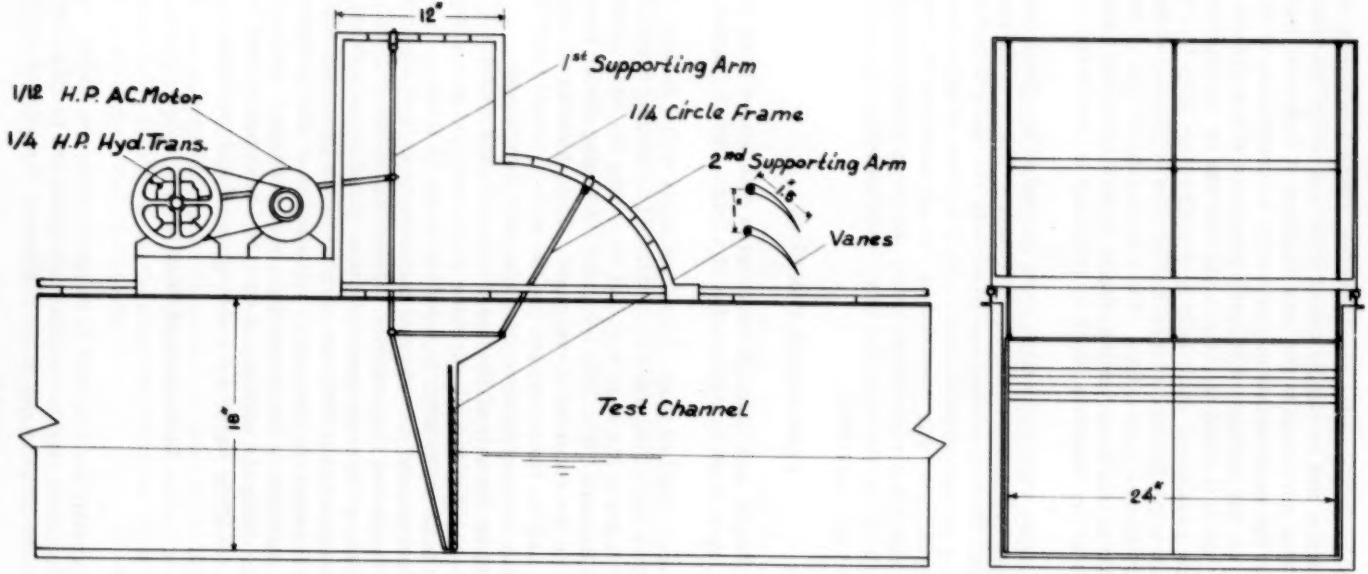
Velocity profiles on the axis of the flume were obtained with a 1/8" diameter pitot tube connected to two manometers which were set at a slope of 45° from the vertical, in order to magnify the differences $\sqrt{2}$ times.

The general experimental procedure in the tests described herein was as follows: The discharge was held constant for varying depths of flow. For each depth the setting of the wave generator was varied several times, and wave heights were recorded at different sections of the flume with the same wave recorder and compared. Necessary adjustments of the generator, such as period and the positions of the connecting rods, were made until a stable wave was obtained. After the stability of wave had been established the characteristics of the resulting waves were photographed and recorded.

Discussion of Results

Stability of Waves

An oscillatory gravity wave is said to be stable when the matter, energy and the phase transported remain constant throughout the propagation of wave. Therefore, instability corresponds to the continuous variation of the foregoing parameters.



Venetian-blind Type of Wave Generator

FIG.1 VENETIAN-BLIND TYPE OF WAVE GENERATOR

It has already been pointed out by various authors that the assumption of constant power transmission is not quite admissible particularly for shallow areas of flat slope and, reductions of as much as 30% in wave heights may occur due to bottom friction in still water. When the media of wave propagation is already in motion, however, additional energy dissipating factors such as internal viscous shear and turbulent mixing shear come into consideration. The waves can remain stable then if only the energy withdrawn from the waves by means of these energy dissipating agencies is exactly balanced with the energy added by the flow. Evidently the propagation of waves alters completely both the velocity distribution and the turbulence structure of the primary flow. The supplied and the dissipated energies which are equal originally, become unbalanced because of the complete alteration of the characteristics of the primary flow and a certain amount of energy, the difference between the supplied and the dissipated energies, is left over. This unbalanced energy is used, in general, either to amplify or to damp the waves. Therefore, from the energy transfer point of view, instability corresponds to an excess or deficiency of the supplied energy with respect to the dissipated energy by various resistances.

The parameters which change the characteristics of the primary flow are essentially the oscillations of the laminar boundary layer due to periodic variations of the pressure, the orbital motion of the water particles due to wave movement and the mass transport and, as a consequence of the latter, a momentum transport in the direction of wave motion. The momentum transported per wave length is simply equal to wave velocity times the mass transported during one wave period. Recent studies have shown that the disturbances on the water surface even of small amplitude, change the mixing length of turbulence considerably. It is an empirically known fact that the turbulence creates a resistive force roughly dependent upon some power of the velocity and indirectly upon some power of the hydraulic radius with a proportionality constant depending upon the stream bed. This empirical formula could perhaps be more rationalized after the determination of the variation of the intensity and scale of turbulence as a function of Froude and Reynolds numbers, so that the wave energy scattered by turbulence can be formulated. Subsequently the attenuation of wave which is at least of the same order of magnitude as the energy scattered can be evaluated.

Because the determination of the stability conditions lacks a sound theoretical solution, the direct approach by way of experiment appears to be the unique method, and may be expected to yield satisfactory results. In order to obtain the conditions of stability of oscillatory gravity waves in water flowing in a smooth channel, the steepness and the ratio d/L of stable waves were determined experimentally and plotted versus the Reynolds number of the primary flow. A series of curves was obtained for constant values of Froude number as is shown in Figs. 2 and 3. Each curve is similar in shape to the one obtained by Yih for the small-amplitude waves in laminar flow. In Yih's solution for each slope of the channel bottom, a hyperbolic type of curve with the arms extending in the directions of the wave number and the Reynolds number separates the stable region from the unstable one. From Figs. 2 and 3, two new graphs were obtained one showing the curves of constant Reynolds numbers for L/d versus Froude number (Fig. 4) and the other showing the surface of neutral stability as a function of L/d, Froude number, and the Reynolds number (Fig. 5). Figures 2, 3, 4, and 5 show clearly that for a pair of given values of the Froude and Reynolds numbers; i. e., for a given flow, there is only one wave which is neutrally stable. For this flow, the other waves are

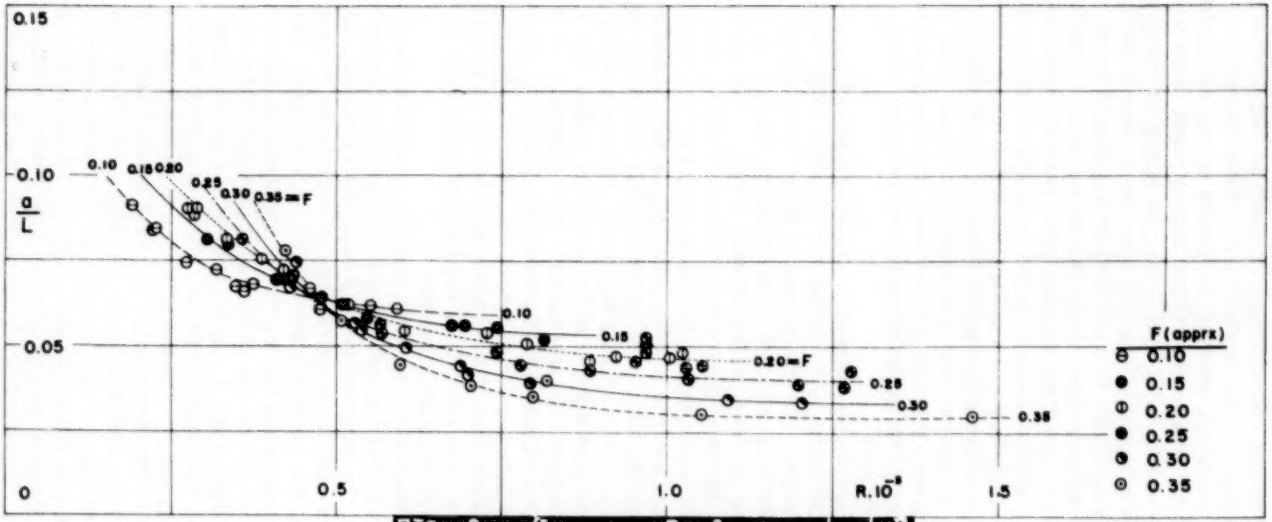


FIG. 2 a/L versus R for constant F

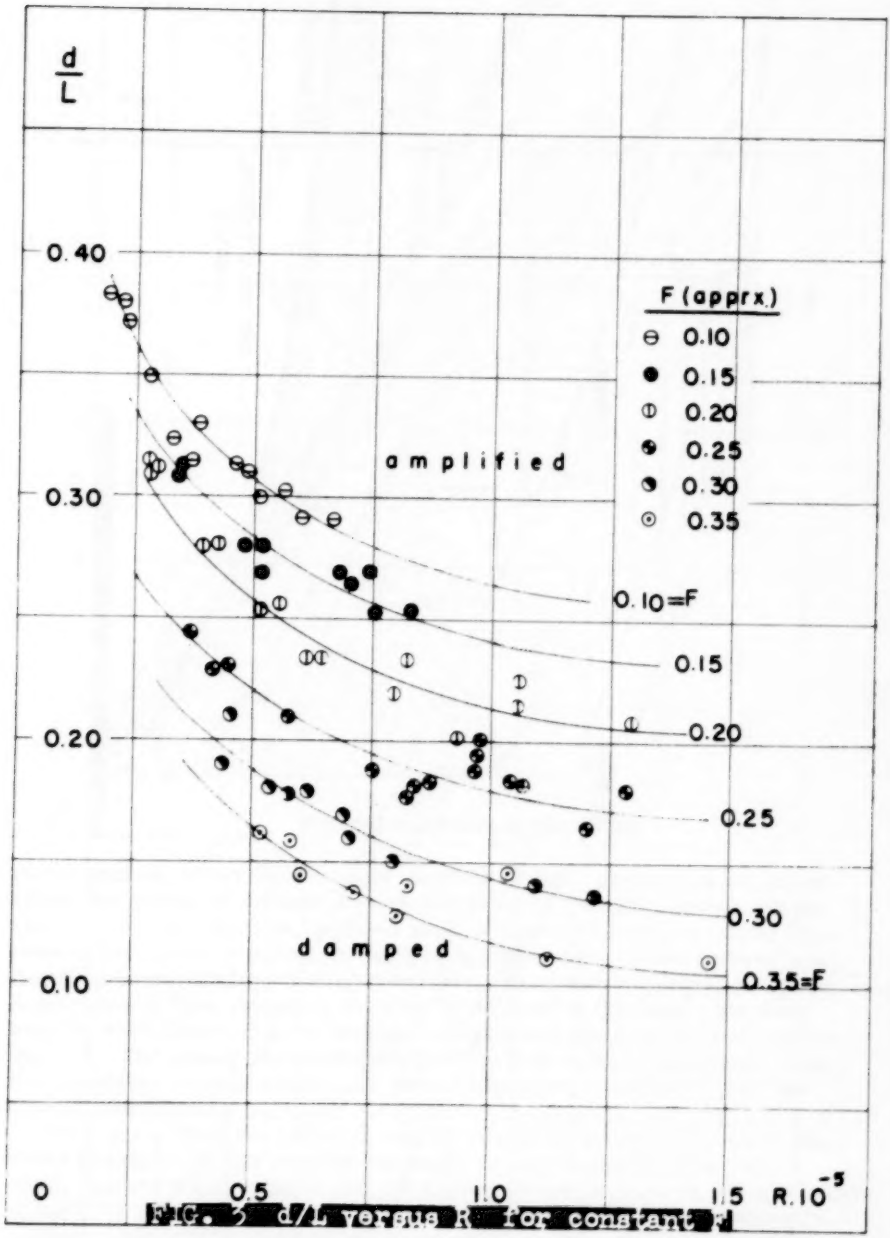


FIG. 2 d/L versus R for constant F

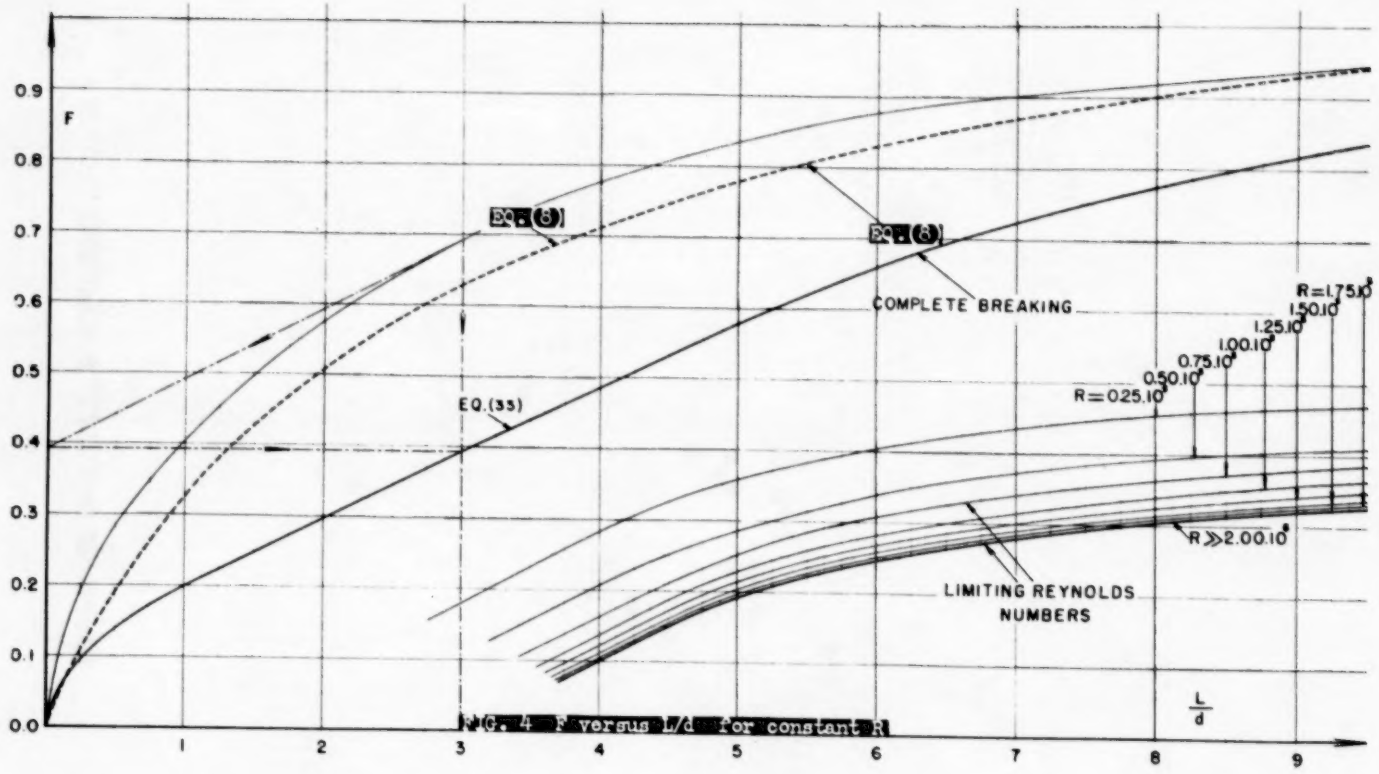
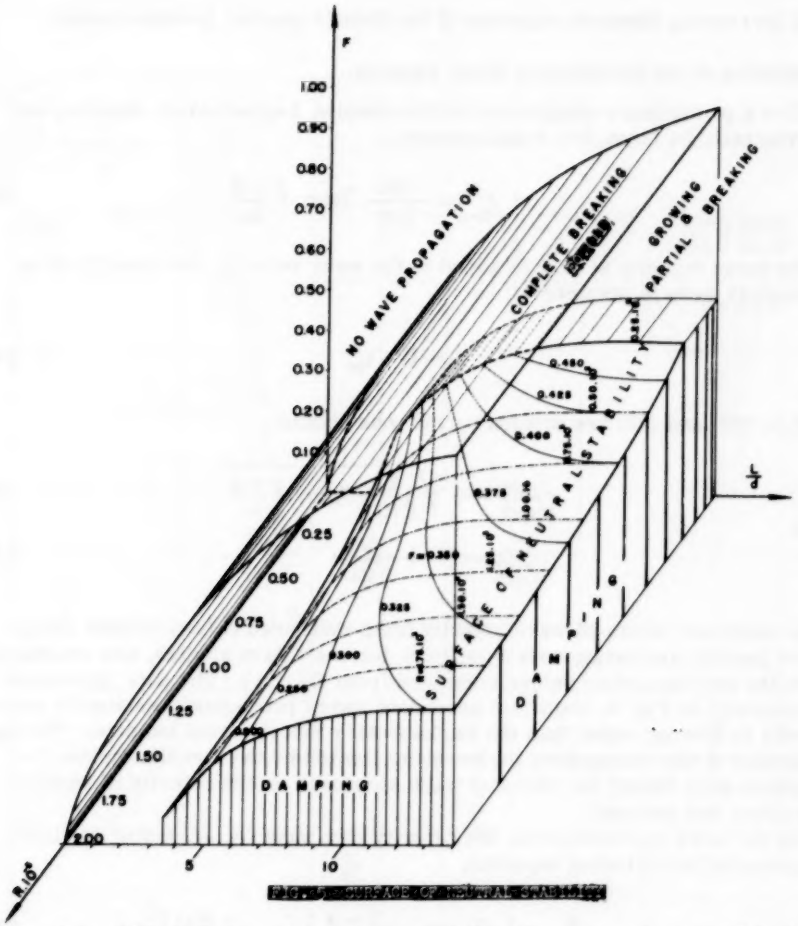


FIG. 4 F versus L/d for constant R



either damped (below the curves of constant Froude numbers) or amplified (above the curves of constant Froude numbers) as they propagate along the channel. A wave which is amplified along its course cannot, obviously grow indefinitely. After a certain distance throughout which the wave grows continuously in amplitude, breaking occurs near the crest, accompanied by the appearance of foam extending down the front slope of the wave. The foam appears to be detached from the mass of the water and rolls over the advancing face. The subsequent development of the wave was not examined. Thus, the possibility of stabilization of a wave by breaking is excluded from the consideration herein.

For a given flow, the ratio d/L and the steepness of the stable wave which would propagate on this current can easily be approximated from Figs. 2 and 3. Figure 2 also reveals that for a given Froude number as the amplitude of the wave increases, the critical Reynolds number for the stable wave decreases. But for large values of the Reynolds number, the steepness of the stable wave depends only on the Froude number. As can be seen clearly from Fig 3, the ratio d/L depends on both the Reynolds and Froude numbers. However, the dependency of the ratio d/L on the Reynolds number diminishes

with increasing Reynolds numbers if the Froude number is kept constant.

Evaluation of the Deviation in Wave Velocity

For a preliminary comparison of the results, Laplace-Airy equation can conveniently be taken into consideration:

$$C^2_{\alpha=0} = \frac{gL}{2\pi} \operatorname{Tanh} \frac{2\pi d}{L} \qquad 19$$

If the mean velocity of flow is added to the wave velocity, the celerity of an ascendant wave is obtained:

$$C_m = U + V_m \qquad 20$$

If Eqs. (19) and (20) are divided by $\sqrt{gd}$, one obtains

$$\frac{C_{\alpha=0}}{\sqrt{gd}} = \sqrt{\frac{L}{2\pi d} \operatorname{Tanh} \frac{2\pi d}{L}} \qquad 21$$

and

$$\frac{C_m}{\sqrt{gd}} = F + \frac{V_m}{\sqrt{gd}} \qquad 22$$

The computed values of wave celerity from measured values of wave length, wave period, and mean velocity of flow, for each wave studied, are compared with the corresponding values computed from Eq. (21). The data, presented graphically in Fig. 6, show that ascendant waves propagate consistently more slowly in flowing water than the Laplace-Airy theory would indicate. The significance of the comparison is, however, limited by the fact that in the Laplace-Airy theory the effect of wave steepness on the celerity of wave is not taken into account.

In the third approximation, the celerity of a wave C_t, according to Struik, is given by the following equation,

$$C_t^2 = \left\{\frac{gL}{2\pi} \operatorname{Tanh} \frac{2\pi d}{L}\right\}\left\{1 - \left(\frac{\pi a}{L}\right)^2 A\right\} \qquad 23$$

in which

$$A = \frac{1}{8}\left\{9 \coth^4 \frac{2\pi d}{L} - 6 \coth^2 \frac{2\pi d}{L} + 5\right\} \qquad 24$$

Thus the velocity is a function of the wave steepness as well as of the ratio d/L. The factor A becomes increasingly important as the ratio d/L becomes smaller. In order to show this importance and to facilitate the computation of C_t, a parameter α, which is defined by the expression

$$\frac{C_t}{C_{\alpha=0}} = 1 + \alpha \qquad 25$$

is introduced into the analysis: The values of α are taken from Fig. 7 for corresponding values of wave steepness and the ratio d/L , and the values of $C_{\alpha=0}$ are computed from Eq. (19). For the same parameters, Fig. 8 gives the values of C_t directly, although less reliably because of the large variation of the celerity with wave steepness.

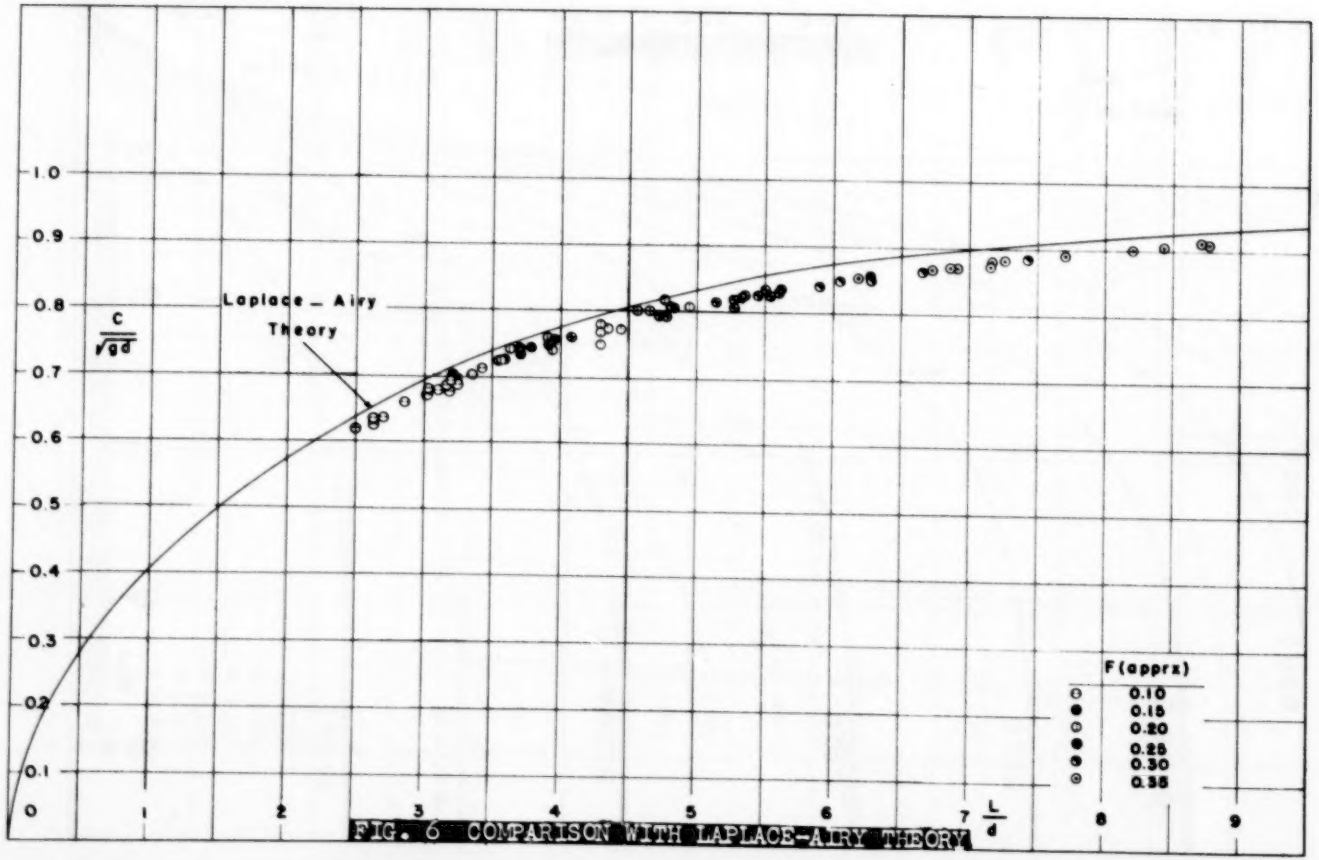


FIG. 6 COMPARISON WITH LAPLACE-AIRY THEORY

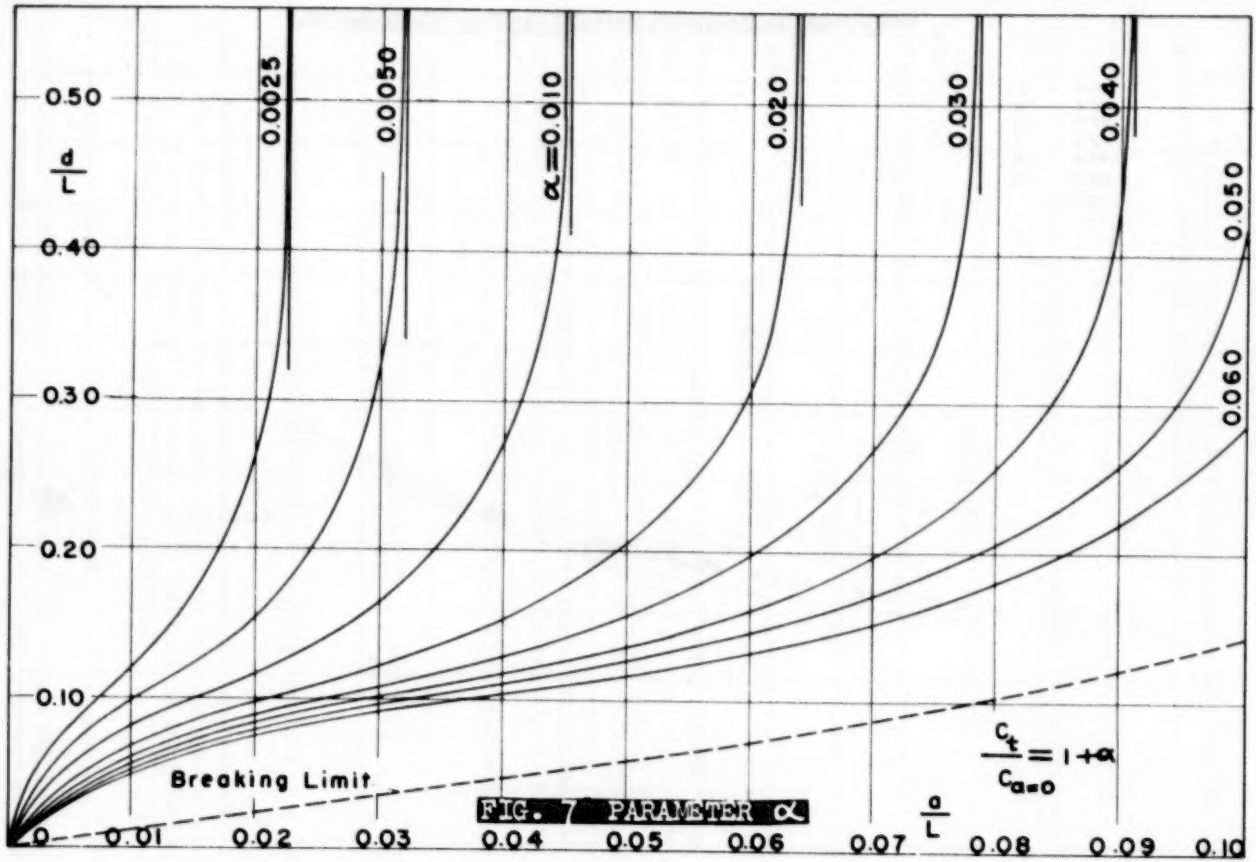


FIG. 7 PARAMETER α

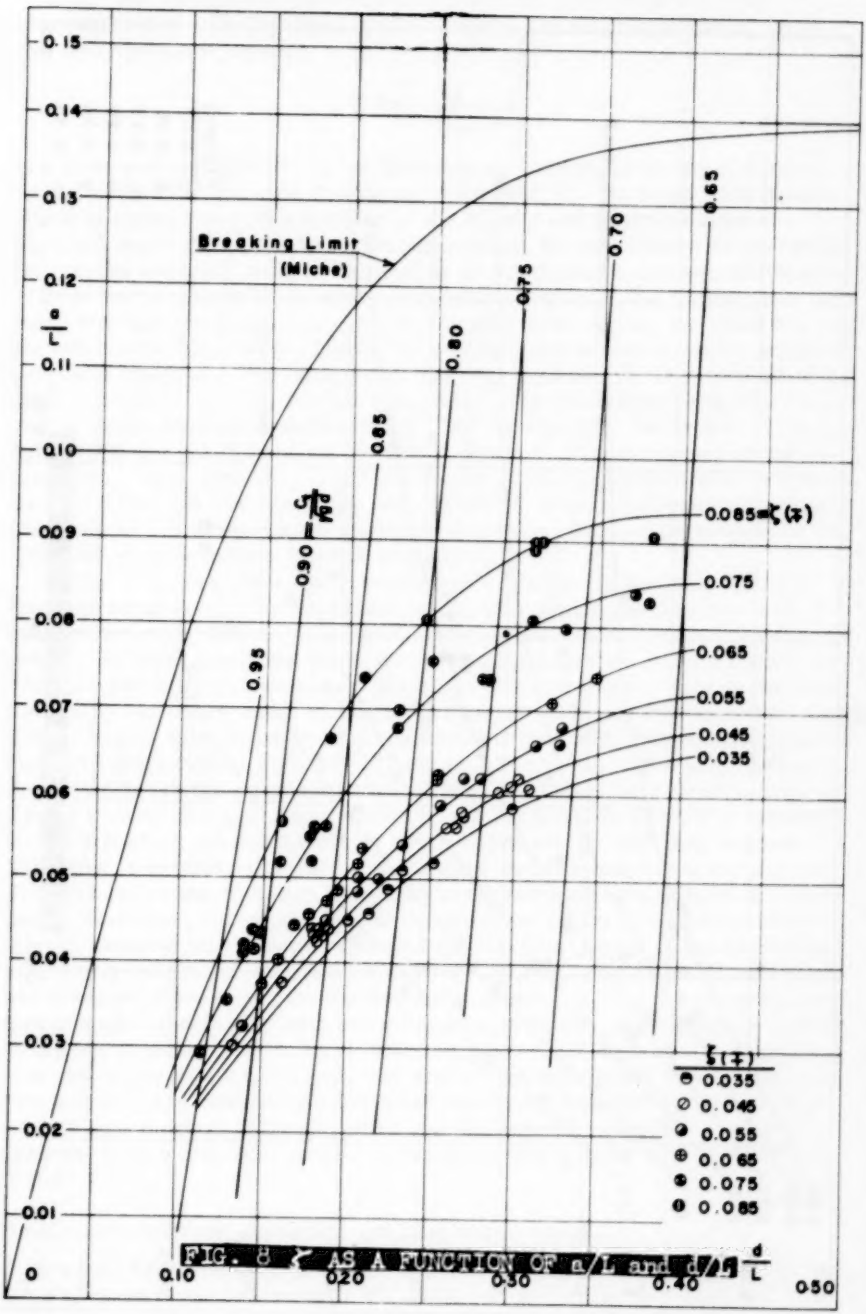


FIG. 8 $\frac{\zeta}{\zeta(x)}$ AS A FUNCTION OF a/L and d/L

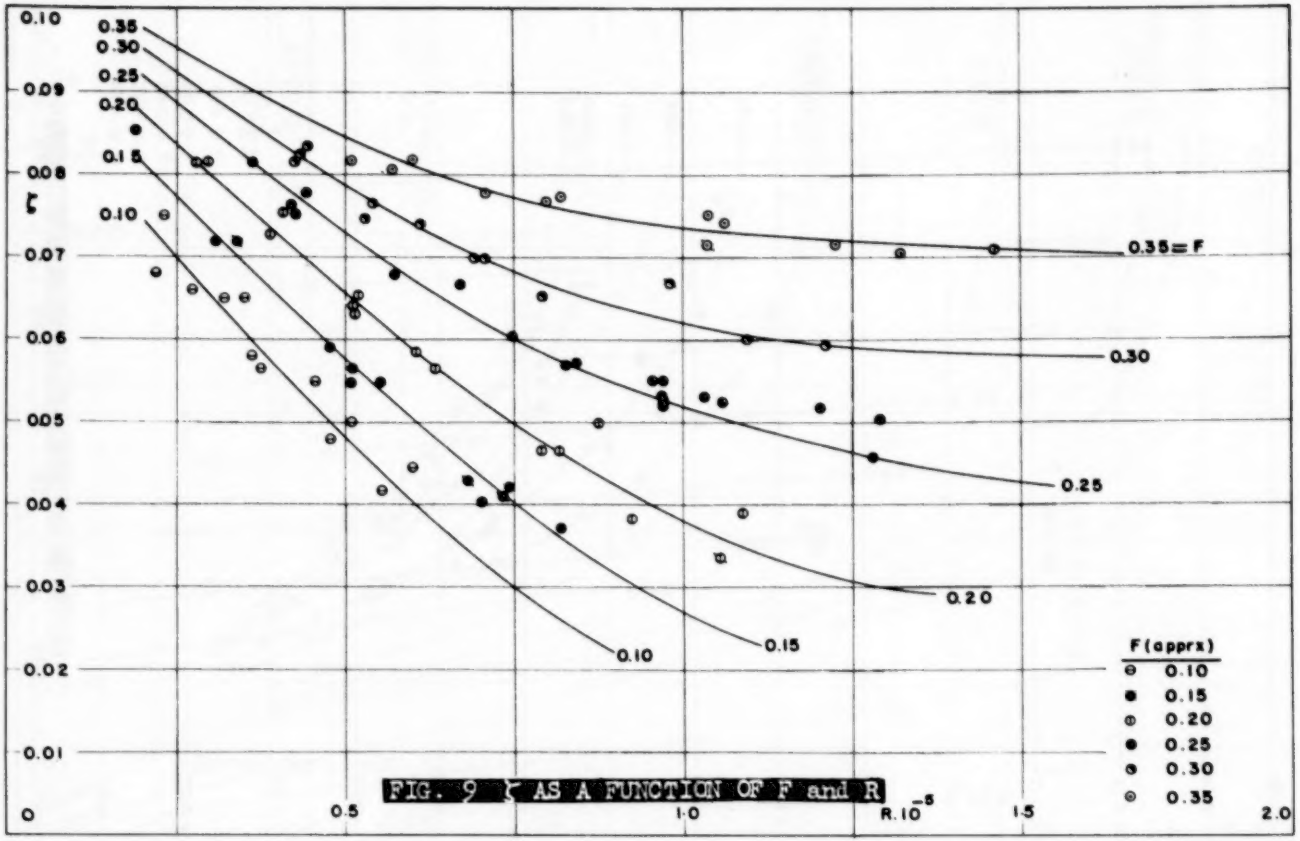


FIG. 9 τ AS A FUNCTION OF F and R

After the fundamental theoretical solution had been properly prepared for comparison with the experimental results, the aforementioned ζ which was defined by the equation

$$\zeta = \frac{V_t}{V_m} - 1$$

was computed and plotted versus Reynolds number and a series of curves were obtained for constant Froude numbers (Fig. 9). As found from dimensional analysis, the ζ is a function of the Froude and Reynolds numbers. Figure 9 shows that for a fixed Froude number, the dependency of the deviation on the Reynolds number diminishes as the Reynolds number increases.

In order to compare the values of ζ determined from the experiments with those obtained from Biésel's analysis for sinusoidal waves, the ratio d/L was plotted versus the Froude number for neutrally stable waves, and a series of curves of constant ζ was obtained in Fig. 10. Equation (5) is presented in the same manner in Fig. 11, for K = 1.34, which was determined from the velocity profiles obtained experimentally. The similarity in the trends of the experimental and the theoretical results is obvious. The agreement is not remarkable. This, however, is not unexpected, since in the analysis turbulence was not taken into consideration, and the waves were of infinitesimal amplitude, and as a consequence of the latter limitation, the mass transport in the direction of wave motion was out of question.

In Fig. 8 the deviation ζ is presented as a function of the ratio d/L and the wave steepness. For the values of d/L less than approximately 0.40, ζ increases rapidly with increasing wave steepness. As the ratio d/L becomes larger, the wave steepness becomes the dominant factor. Here it should be remembered that the foregoing comparison and conclusions have so far been made only for experiments in which the channel walls and bottom were smooth. For a channel with an artificially roughened bottom and smooth sides, similar plots are presented in Figs. 12, 13, and 14. These figures clearly show that the deviation for the roughened channel is greater than the corresponding one for the smooth channel. Unfortunately, it is not possible to draw a definite conclusion about the dependency of the deviation on the Reynolds number. Moreover, a careful examination of Figs. 14 and 15 reveals that for a given Froude number the deviation does not vary greatly with the degree of roughness. Therefore, the dependency of the deviation on the Reynolds number is slight. However, the question remains open as to whether, in the case of an artificially roughened channel, the deviation is a function of the Froude number alone or depends also on the Reynolds number. This point could be clarified through experiments with new roughness elements geometrically similar to the one previously used but of different size. Such experiments would enable one to prepare a family of plots with a constant Froude number for each plot and each one containing a series of curves for constant relative roughnesses as a function of the deviation and the Reynolds number. Figure 15 presents such a family of curves, although incomplete, for a Froude number of 0.30.

Relationship between Ascendant and Descendant Waves

If waves are generated in a fluid which is already in motion by placing the wave generator somewhere near the middle of the flume, one observes that descendant waves are much longer than ascendant ones whereas their period is equal.

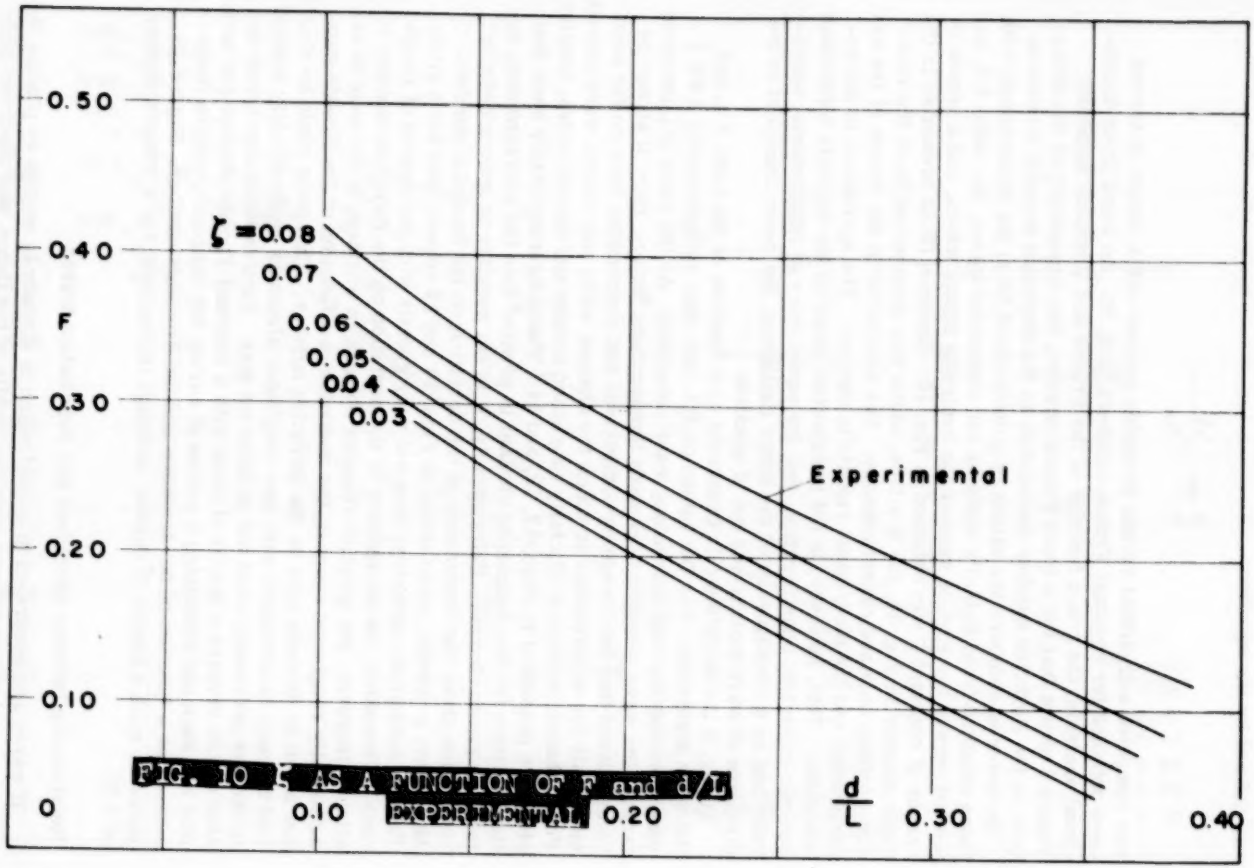


FIG. 10 ζ AS A FUNCTION OF F and d/L EXPERIMENTAL

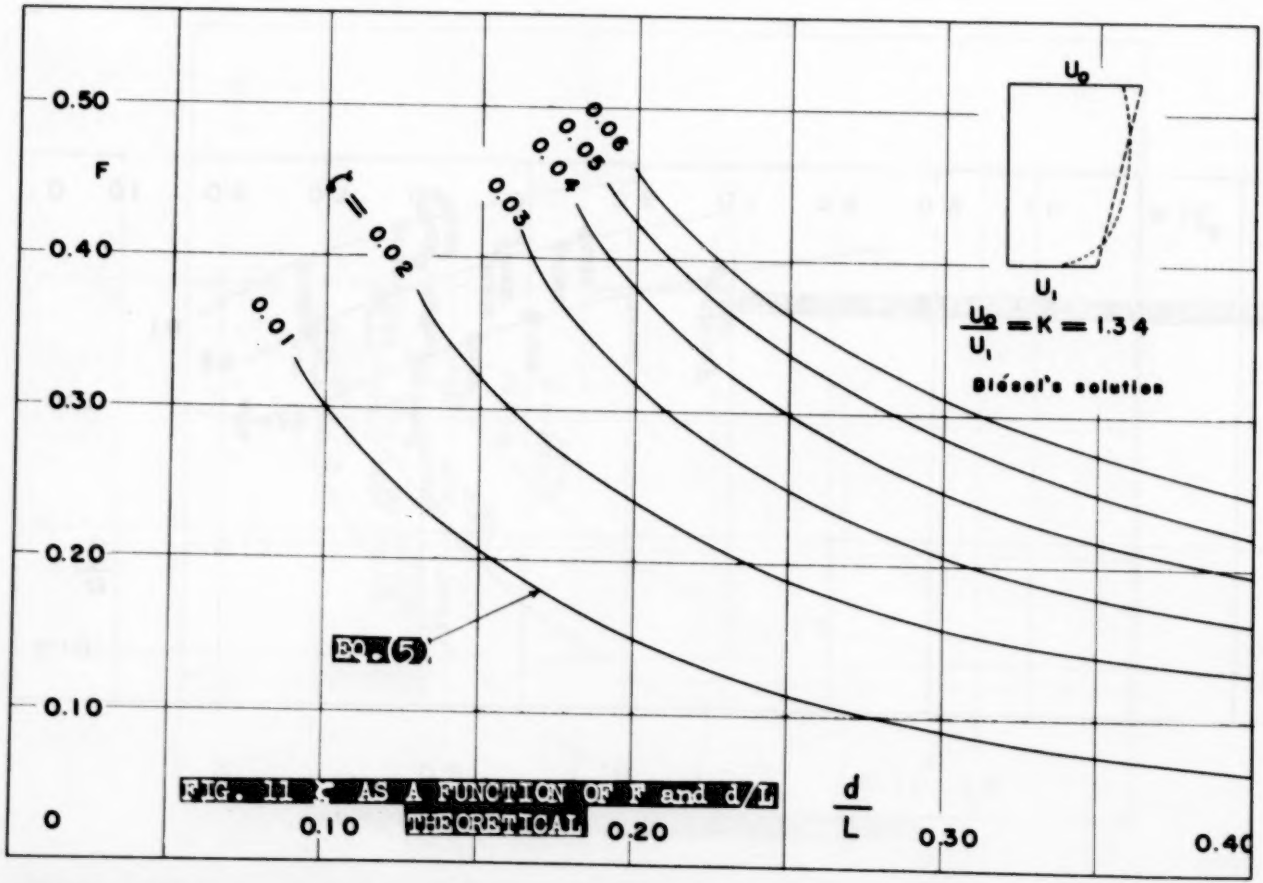


FIG. 11 ε AS A FUNCTION OF F and d/L

FIG. 12 a/L Versus R For d/t Constant
R.10^5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
a/L
0.10
0.05
d/t=25
20
16
0.294=F
0.294
0.262
0.236
0.232
0.216
0.185
0.178
0.185
0.164
0.159=F
0.158
0.158
0.158
0.136
0.134
0.213
0.214
0.176
0.176
0.177
0.234

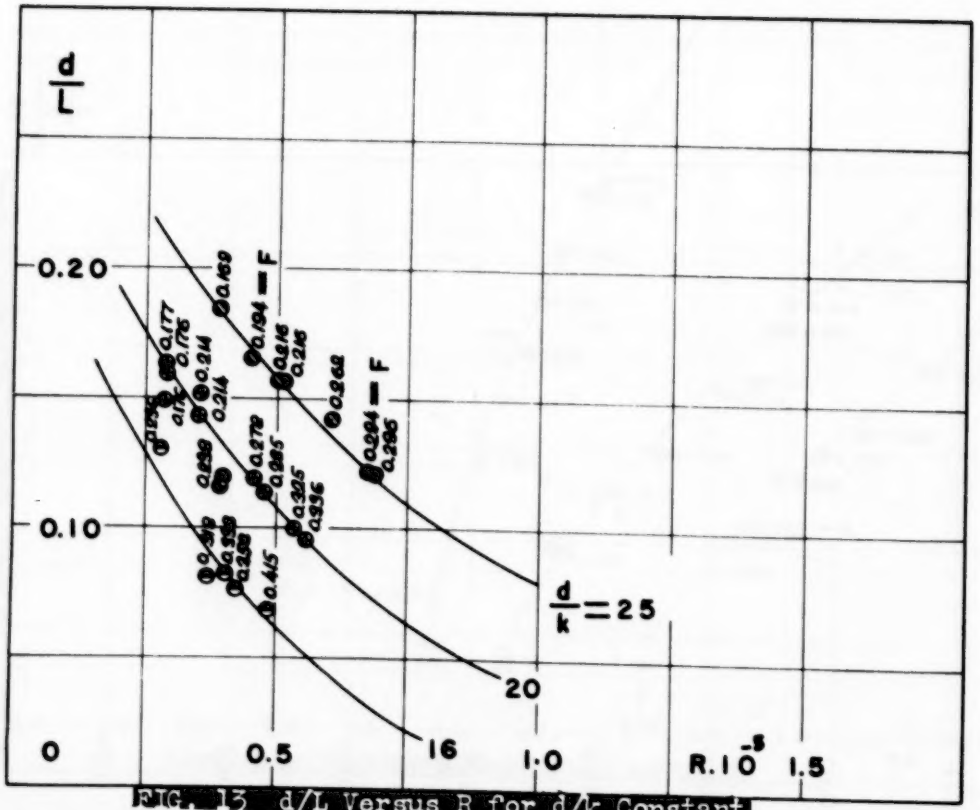


FIG. 13 d/L Versus R for d/k Constant

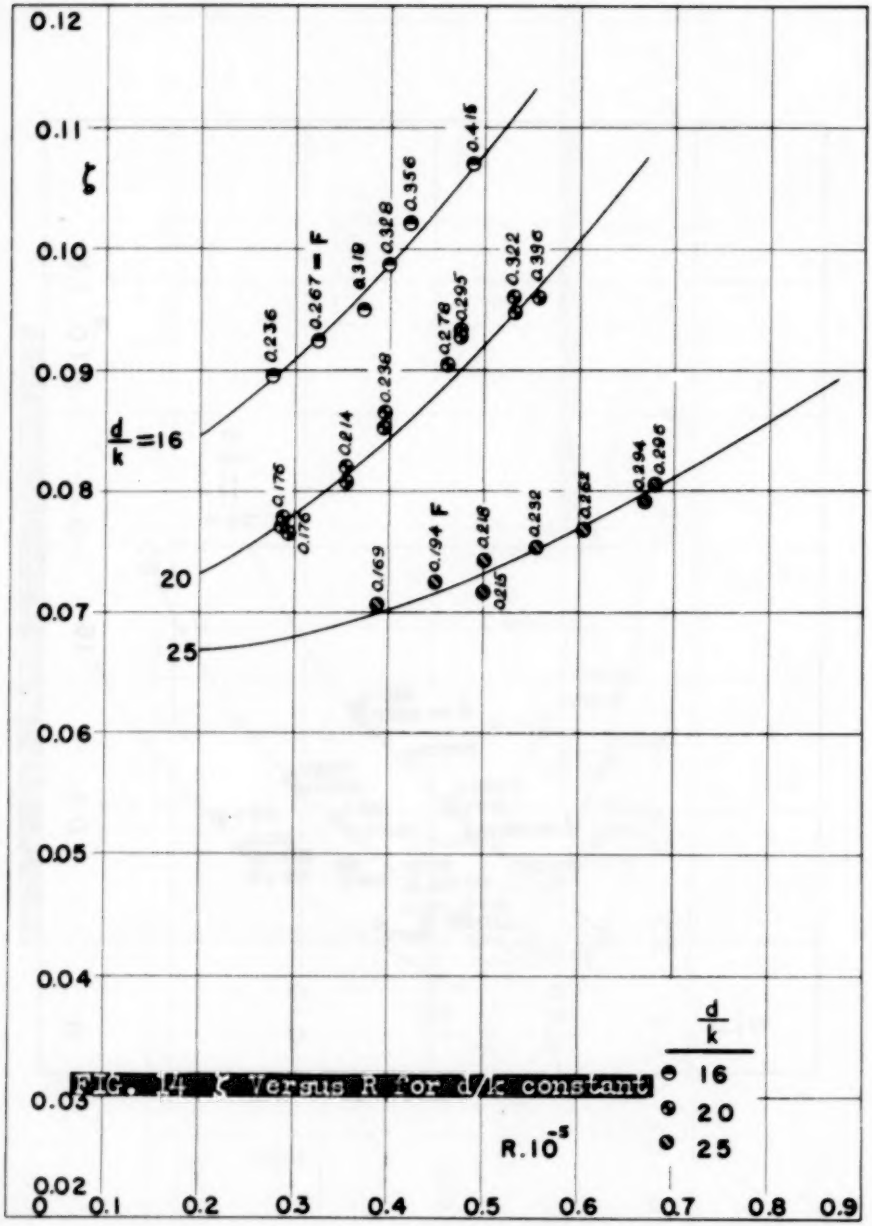


FIG. 14 ζ Versus R for d/k constant

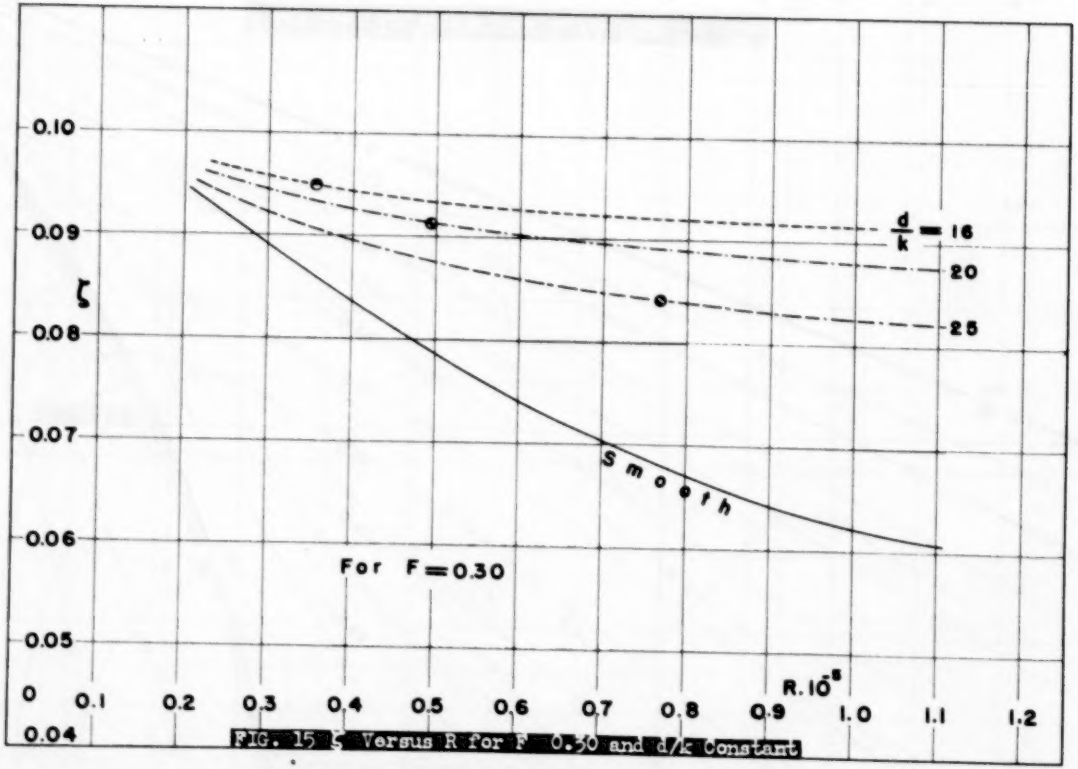


FIG. 15 ζ Versus R for F 0.30 and d/k Constant

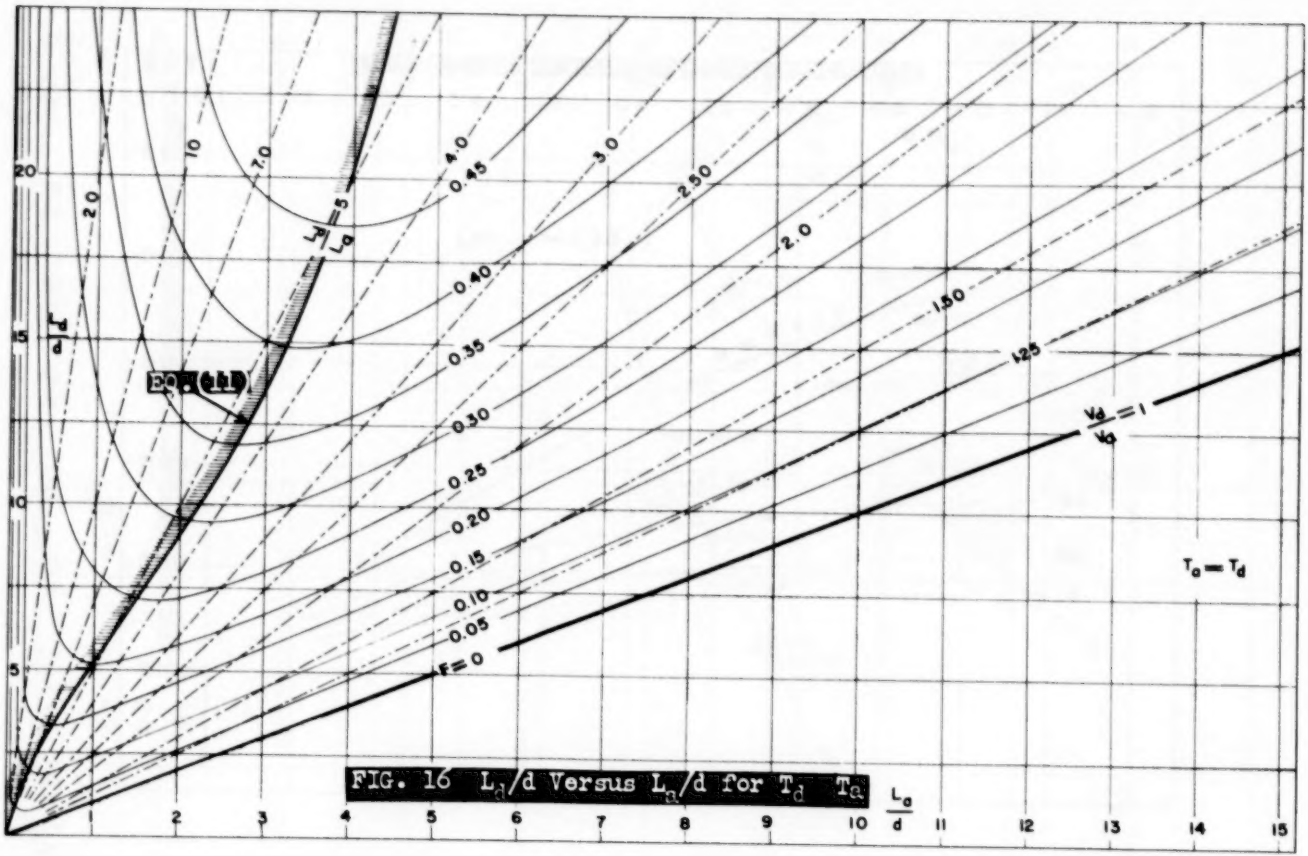


FIG. 16 L_d/d Versus L_a/d for T_d T_a

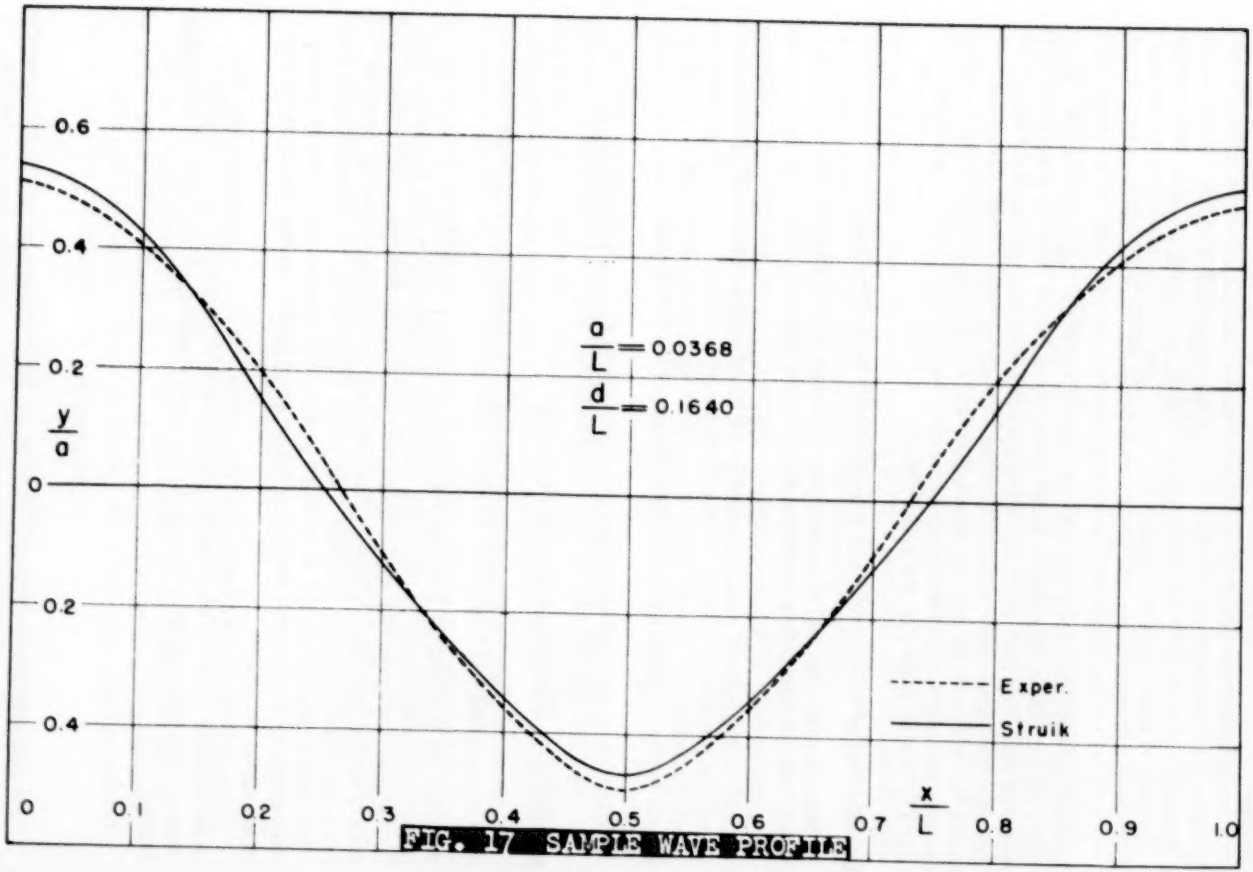


FIG. 17 SAMPLE WAVE PROFILE

According to the Laplace-Airy solution, the ratio of the lengths of an ascendant and a descendant wave of the same period can be written as follows:

$$\frac{\frac{L_d}{d}}{\frac{L_a}{d}} = \frac{\sqrt{\frac{L_d}{2\pi d}\tanh\frac{2\pi d}{L_d}} + F}{\sqrt{\frac{L_a}{2\pi d}\tanh\frac{2\pi d}{L_a}} - F} \qquad 25$$

Constant Froude numbers are plotted versus L_d/d and L_a/d in Fig. 16. This figure shows that a descendant wave propagates faster than an ascendant wave provided that they both have the same period and travel in the same flow. This fact becomes more apparent as the Froude number is increased and shows the difficulty of testing descendant waves in a short test channel. The limiting barrier beyond which no ascendant wave propagation is possible (Eq. 11) is also shown in the same figure.

Comparison of Theoretical and Experimental Wave Profiles

Typical experimental profiles selected from the numerous runs are compared with the theoretical profiles given by the equation (3):

$$\eta = \frac{1}{2}m\delta^2\coth md + \left\{1 + \frac{3}{8}m^2\delta^2\tanh^2 md\left(1 + 2\coth^2 md\right)\right\}\delta\cos mx$$
$$+ \frac{1}{4}m\delta\coth md\left(3\coth^2 md - 1\right)\delta\cos 2mx$$
$$+ \frac{3}{64}m^2\delta^2\left(3\coth^2 md - 1\right)\left(3\coth^4 md + 1\right)\delta\cos 3mx \qquad 26$$

in which δ is determined from the equation:

$$a = 2\delta\left\{1 + \frac{3}{64}m^2\delta^2\left(9\coth^6 md + 13\coth^4 md + 11\coth^2 md - 1\right)\right\} \qquad 27$$

Although no theoretical function represents the wave profile exactly for all possible combinations of ratio d/L and wave steepness, it was found that the above equation gives the most consistent agreement between observations and theory within the range tested. The deviation of the experimental profiles from the theory was as follows: For large amplitudes, the experimental profiles have a sharper crest with more gentle side slopes and a flatter trough than given by the theory, while the lower amplitudes exhibit characteristics of an opposite nature.

Thus, for the amplitudes and depths examined in this investigation, the experimental profiles vary about the theoretical profile as described by Struik. The wave profile in Fig. 17 is representative of the numerous profiles tested.

CONCLUSIONS

The detailed laboratory and theoretical investigation of oscillatory gravity waves in flowing water described herein produced results from which the following conclusions can be drawn:

(1) For a given flow, there is only one neutrally stable wave. All the other waves for this flow show either an exponential damping or an amplification during their propagation along the channel. It is readily possible to separate the two regions by the surface of neutral stability which was determined experimentally and presented graphically. In view of the complexity of stability phenomena, it is no wonder that there is considerable difficulty in finding a reasonable solution for the stability of oscillatory gravity waves in flowing water. This author has no basic solution to present and leaves the theoretical determination of this elusive problem to future investigators.

(2) The celerity of a stable ascendant wave in flowing water is up to 10% smaller than that obtained from the theories of Stokes and Struik. This finding is contrary to the commonly held assumption that the characteristics of waves propagated in flowing water are identical with those propagated in still water. Thus, the rule of superimposition is not permissible. Two present theories are found to be inadequate for the waves in flowing water.

The deviation of the measured wave velocity from that obtained from potential theory increases with increasing Froude numbers and decreases with increasing Reynolds numbers. The larger the Reynolds number is, the smaller is the dependency of the deviation on the Reynolds number.

(3) For a channel with an artificially roughened bottom, the deviation is greater than the corresponding one in a channel with a smooth bottom. For example, for a Froude number 0.30 and Reynolds number 10^5, deviation is about 6% for a smooth channel and about 9% for a rough channel. For rough channels, however, no conclusion can be drawn as to the dependency of the deviation on the Reynolds number because of experimental limitations. Indications are such that the Reynolds number has either little or no effect on the deviation.

(4) The experimentally determined profiles of stable ascendant oscillatory gravity waves vary about the theoretical profiles as described by Struik. It was found that Struik's profile equation gives the most consistent agreement between observations and theory within the range tested.

(5) The hydraulic engineer who finds himself concerned with the problems of oscillatory gravity waves in flowing water should consider this paper as but a basic introduction to a body of knowledge and scant available literature that exists on wave motion in flowing water. There remain multitudes of unsolved problems of both a theoretical and a practical nature, but a high degree of understanding of the mechanics involved and an appreciation of the importance of sound theoretical concepts must be brought to their solution.

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TABLE OF SYMBOLS

a wave amplitude
b width of flume

C	celerity of wave
C_m, C_t	measured and theoretical celerities
d	depth of flow
L	wave length
α/L	wave steepness
d/L	ratio d/L
L/d	relative length of wave
F	$\bar{U}/\sqrt{gd}$, Froude number
g	gravitational acceleration
T	wave period
U_0, U_1	surface and bottom velocities of primary flow
V	velocity of wave
m	$2\pi/L$
V_m, V_t	measured and theoretical wave velocities
$\bar{U}$	mean velocity of flow
α	a parameter
β	channel shape factor
ζ	$\frac{V_t - V_m}{V_m}$, deviation
λ	spacing of roughness elements
k	height of roughness element
w	width of roughness element
d/k	relative roughness
λ/w	relative spacing of roughness elements

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